

# On spectral multiplicities of Gaussian actions

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## Abstract

For any set  $M \subset \mathbf{N}$  there are mixing Gaussian automorphisms and non-mixing Gaussian automorphisms (as well as automorphisms which are disjoint from any Gaussian action) with continuous spectrum for which  $M \cup \{\infty\}$  is the set of their spectral multiplicities. We show also that for a Gaussian flow  $\{G_t\}$  the spectral multiplicities of  $G_t$ ,  $t > 0$ , could be different.

## 1 Spectral multiplicities of some Gaussian automorphisms

Let  $Exp(U) = \oplus_{n=1}^{\infty} U^{\odot n}$ , where  $U$  is a **unitary operator with simple continuous singular** spectrum.

**Lemma.** *There are  $z_m$ ,  $m \in \mathbf{N}$ ,  $|z_m| = 1$ , such that the operators  $z_m U$  are pairwise spectrally disjoint.*

(Proof.  $U$  has singular spectrum = its spectral measure  $\sigma$  is singular. Let  $\sigma_z$ ,  $|z| = 1$ , is defined by the condition

$$\sigma_z(A) = \sigma(zA).$$

The measure  $\sigma$  is singular iff for almost all  $z$  with respect to Lebesgue measure on the unit circle the measures  $\sigma_z$  and  $\sigma$  are disjoint.)

Denote  $\mathbf{m}U = \oplus_{n=1}^m U$ . Given  $M \subset \mathbf{N}$ , we set  $V = \oplus_{m \in M} \mathbf{m}(z_m U)$ , where all  $z_m$  are from Lemma.

**Assertion 1.** *If a) the product  $U^{\odot 2}$  has absolutely continuous spectrum, or*

*b) the operators  $U^{\odot n}$  have homogeneous spectra of infinite multiplicity as  $n > 1$ , and all operators  $U^{\odot n}$  are pairwise disjoint,*

*then the set  $\mathcal{M}(Exp(V))$  of spectral multiplicities of the operator  $Exp(V)$  is*

$$M \cup \{\infty\}.$$

Proof. It is not hard to see that

$$\text{Exp}(V) = V \oplus \bigoplus_{i=1}^{\infty} W_{(i)},$$

where in a) the operator  $W$  has Lebesgue spectrum, and in b)  $W$  has singular spectrum.

**Realization of a).** The examples of singular measures  $\sigma$  (considered as a spectral measure for a unitary operator) with the absolutely continuous convolution  $\sigma * \sigma$  are classical (Schaeffer-Salem measures, 1939-1943, see [2]).

**Realization of b).** There is an operator  $U$  of simple spectrum such that

$2U^{n_i} \rightarrow_w I$  (providing the operators  $U^{\odot n}$  to be pairwise disjoint), and

$U^3$  is spectrally isomorphic to  $U \oplus U \oplus U$  (this implies that the operators  $U^{\odot n}$  have homogeneous spectra of infinite multiplicity as  $n > 1$ ).

We apply here the operator induced by an exponential self-similar 2-adic Chacon map, see arXiv:1311.4524.

There is well-known correspondence between a unitary operator  $U$  with continuous spectrum and an ergodic Gaussian automorphism (of a probability space)  $G(U)$  which is spectrally isomorphic to the operator

$$\bigoplus_{n=0}^{\infty} U^{\odot n},$$

where  $U^{\odot 0} = 1$  denotes one-dimension identity operator (acting on the space of constants).

Thus, from the above facts we get the following result, adding a string to the list of realizable sets (the sets of spectral multiplicities for ergodic automorphisms).

**Theorem 1.** *For any set  $M \subset \mathbf{N}$  there are mixing Gaussian automorphisms and non-mixing automorphisms with singular spectrum for which  $M \cup \{\infty\}$  is the set of their spectral multiplicities.*

## Spectral multiplicities of non-Gaussian automorphisms.

**Theorem 1.1** *For any set  $M \subset \mathbf{N}$  there is a weakly mixing automorphism  $T$  which is disjoint with any Gaussian automorphism and  $\mathcal{M}(T) = M \cup \{\infty\}$ .*

For this we consider  $T = T_1 \times T_2 \times \dots$ , where  $T_i$  have homogeneous spectra with multiplicities from  $M$  and all pairwise convolutions of their spectral measures are Lebesgue.

We recall that all sets in the form  $M \cup \{1\}$  and  $M \cup \{2\}$  are realizable via special skew products (Ageev, Danilenko, Katok, Kwiatkowski, Lemanczyk et al, see [1]).

## 2 Spectral multiplicities of automorphisms in the Gaussian flow

Some results on non-usual multiplicities of powers for a weakly mixing automorphism see in [3]. Any Gaussian automorphism is embedded in a Gaussian flow. We prove that *there are Gaussian flows  $\{G_t\}$  with non-constant sets  $\mathcal{M}(G_t)$  (of the spectral multiplicities for automorphisms  $G_t$ ,  $t > 0$ ).*

We consider an operator  $U = V \oplus (-V)$  with simple spectrum, where  $V$  is disjoint with the powers  $V^{\odot n}$ ,  $n \geq 2$ , and the corresponding Gaussian flow  $G_t$ ,  $G_1 = G(U)$ . Then we get <sup>1</sup>

**Theorem 2.** *There is  $U$  such that  $\mathcal{M}(G(U)) = \{1, \dots\}$  and  $\mathcal{M}(G(U^2)) = \{2, \dots\} \subset 2\mathbf{N}$ .*

Now we change an operator setting  $U$  as in section 1, the case b. The operator  $U$  has simple spectrum (it is induced by a rank-one transformation). Its power  $U^{3^k}$  is isomorphic to the direct sum of  $3^k$  copies of  $U$ . All symmetric powers  $U^{\odot 2}, \dots$ , have infinite spectral multiplicity. Thus, we obtain

**Theorem 3.** *There is  $U$  such that  $\mathcal{M}(G(U^{3^k})) = \{3^k, \infty\}$ .*

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<sup>1</sup>This answers the question 11 from [1]

However a general simple fact shows that  $\mathcal{M}(G_t)$  is constant almost everywhere.

**Assertion 2.** *Let  $\{U_t\}$  be a unitary flow with simple singular spectrum. Then for almost all  $t$  the operators  $U_t$  have simple spectra as well. Generally, if the spectral measure of a flow is singular, then  $\mathcal{M}(U_t) = \mathcal{M}(\{U_t\})$  for almost all  $t$ .*

By use of combinations of operators in the form  $\oplus_{m=1}^n z_m U$  we get a variety of flows with non-constant multiplicity function  $\mathcal{M}(G_t)$ .

**Theorem 4.** 1. *For any prime  $p$  and any  $m < p$  there is a Gaussian flow  $\{G_t\}$  such that  $\mathcal{M}(G_1) = \{1, \infty\}$  and  $\mathcal{M}(G_p) = \{m, \infty\}$ .*

2. *There is a Gaussian flow  $\{G_t\}$  such that for any finite integer set  $M$  for some  $t = t(M) > 0$   $\mathcal{M}(G_t) = \{1, \infty\} \cup M$ .*

3. *For any set  $M \subset \mathbf{N}$  there is a Gaussian flow  $\{G_t\}$  with singular spectrum such that  $\mathcal{M}(\{G_t\}) = \{1, \infty\}$ ,  $\mathcal{M}(G_1) = M \cup \{\infty\}$ , and any interval  $I \subset \mathbf{R}$  contains an infinite set  $\{t_k\}$  for which multiplicity sets  $\mathcal{M}(G_{t_k})$  are pairwise different.*

4. *Let  $P$  denote the set of primes. For any function  $m : P \rightarrow \mathbf{N}$ ,  $m(p) \leq p$ , there is a Gaussian flow  $\{G_t\}$  such that*

$$\mathcal{M}(G_n) = \{\infty\} \cup \{m(p_1), m(p_2), \dots, m(p_k)\},$$

*where  $n = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$ , the factorization of  $n$  into a product of primes.*

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## References

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